Lecture 13: Differentials

Note Title 10/27/2019

Kähler differentials

A: ring, B = A-algebra, M = B-module

Definition: An A-derivation of B into M is a map $d: B \longrightarrow M$ s.t.

① d is additive

② $d(bb') = b(db') + b(db')$

③ $da = 0$, $\forall a \in A$

Definition: $\Omega_{B/A} = \{db \mid b \in B\} / \sim$

module of relative differential forms of B over A

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, dashed, "\exists f"] \\ & M \end{tikzcd}$$

w/ universal property

Proposition 1: $0 \to I \to B \otimes_A B \xrightarrow{f} B \to 0$

$(b, b') \longmapsto bb'$

as B-module by multiplication on the left

then $I/I^2 \cong \Omega_{B/A}$, w/ $d: B \longrightarrow I/I^2$

$b \longmapsto \underbrace{1 \otimes b - b \otimes 1}$ generators of I?

pf: • $d': B \longrightarrow I/I^2$ is a derivation

$b \longmapsto 1 \otimes b - b \otimes 1$

$$d(bb') = 1\otimes bb' - bb'\otimes 1 \overset{?}{=} b(d'b') + b'(db)$$
$$\qquad\qquad\qquad \| $$
$$b(1\otimes b' - b'\otimes 1) + b'(1\otimes b - b\otimes 1)$$

$$I^2 \ni \underbrace{(1\otimes b - b\otimes 1)\otimes(1\otimes b' - b'\otimes 1)}_{\text{generators of } I^2} = 1\otimes bb' + bb'\otimes 1 - b'\otimes b - b\otimes b'$$

$/I^2 \sim d'$ satisfies Liebiniz rule .

ex. $B = A[x_1, \cdots, x_n]$, $\quad \Omega_{B/A} = \bigoplus_{i=1}^{n} B\, dx_i$ free B-module

② reduces df to linear combination of dx_i $/B$

Sheaves of Differentials

$$f: X \longrightarrow Y \quad \rightsquigarrow \quad \begin{tikzcd} \Delta : X \arrow[r, "\cong"] & \Delta(X) \arrow[r] \arrow[dr, hook, "\text{closed}"'] & X \times_Y X \\ & & W \arrow[u, hook, "\text{open}"'] \end{tikzcd}$$

$\mathcal{I} :=$ ideal sheaf of $\Delta(X) \hookrightarrow W$

Definition: $\Omega_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ sheaf of relative differentials of X over Y

$U = \operatorname{Spec} A \underset{\text{open}}{\subseteq} Y$, $\quad V = \operatorname{Spec} B \subseteq X \quad$ s.t $\quad f(V) \subseteq U$

then $\quad V \times_U V \subseteq X \times_Y X$ affine open $\qquad \Delta(X) \cap V \times_U V \hookrightarrow V \times_U V$ w/ ideal sheaf I

$$V\times_U V \cong \operatorname{Spec}(B \otimes_A B)$$

$$\mathcal{I}\big|_{\Delta(X)\cap V\times_U V} \cong \widetilde{I}$$

$$\Omega_{X/Y}\Big|_{\Delta(X)\cap V\times_U V} \cong \widetilde{I/I^2} \cong \widetilde{\Omega_{B/A}}$$

$\Omega_{X/Y}$ quasi-coherent

Proposition 2:
$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$
then $\Omega_{X'/Y'} = g'^{*}\Omega_{X/Y}$

pf: (Local version) A', B A-algebras $B' = A' \otimes_A B$

then $\Omega_{B'/A'} = \Omega_{B/A} \otimes B'$

$$\begin{tikzcd} \Omega_{B'/A'} \arrow[r, shift left, "\exists !"] & B' \otimes_B \Omega_{B/A} \arrow[l, shift left] & \Omega_{B/A} \arrow[l] \arrow[ll, bend right, "\exists !"'] \\ & B' = A' \otimes_A B \arrow[ul, "d'"] \arrow[u, "d''"'] & B \arrow[l, "i"'] \arrow[u, "d"'] \\ & A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

M : B-module, M': B'-module

$\mathrm{Hom}_B(M, M') = \mathrm{Hom}_{B'}(M \otimes_B B', M')$

$B \xrightarrow{i} B' \longrightarrow \Omega_{B'/A'}$ is a A-derivation

$B' \longrightarrow B' \otimes_B \Omega_{B/A}$ is a A'-derivation

$$d''(a' \otimes 1) = a' \otimes d(\underset{A}{1}) = 0$$

$$\begin{aligned} d''\big((a_1' \otimes b_1)\cdot(a_2' \otimes b_2)\big) = d''(a_1'a_2' \otimes b_1 b_2) &= a_1'a_2' \otimes d(b_1 b_2) \\ &= a_1'a_2' \otimes (b_2 db_1 + \underline{b_1 db_2}) \\ &= \underline{(a_1' \otimes b_1) \otimes a_2' db_2} + (a_2' \otimes b_2) \otimes a_1' db_1 \end{aligned}$$

In particular, S multiplicative system of B

$$S^{-1}\Omega_{B/A} \cong S^{-1}A \otimes \Omega_{B/A} \cong \Omega_{S^{-1}B/A}$$

$$\bar{d}\left(\frac{b}{s}\right) = \frac{db}{s} - \frac{b\,ds}{s^2}$$

Proposition 3. $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\Rightarrow \quad f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

pf: (local version) $A \longrightarrow B \xrightarrow{\psi} C$ ring homomorphism

then $\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$

$d_{B/A}(b) \otimes c \longmapsto c\, d_{C/A}(\psi(b))$

$d_{C/A}(c) \longmapsto d_{C/B}(c)$

Proposition 4. $f: X \longrightarrow Y$

$\uparrow$ closed w/ ideal sheaf $\mathscr{I}$

Z

then $\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0$

pf (local version) B: A-algebra, $I \triangleleft B$, $C = B/I$

then $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes C \longrightarrow \Omega_{C/A} \longrightarrow 0$ as C-module

naturally C-module

$\bar{b} \longmapsto d_{B/A}(b) \otimes 1$

$b \in I$

$d_{B/A}(b) \otimes c \longmapsto c\, d_{C/A}(b+I)$

Theorem 1. (Euler sequence)

$Y = \operatorname{Spec} A, \quad X = \mathbb{P}^n_A$

$$\Rightarrow \quad 0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

dually, $\quad 0 \to \mathcal{O}_X \to \mathcal{O}(1)^{\oplus n+1} \to T_{X/Y} \to 0$

relative tangent sheaf

In case of $\mathbb{P}^1$, $\quad 0 \to \underset{\underset{\mathcal{O}(-2)}{\cong}}{\Omega_{\mathbb{P}^1}} \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0$ $\quad \because \deg \Omega_{\mathbb{P}^1} = -2$

pf: $S = A[x_0, \cdots, x_n]$, $\quad E = S(-1)^{\oplus n+1}$

w/ basis $e_0, \cdots, e_n$ of degree 1

i.e $e_i = (0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0)$

$$0 \to M \to E \to S \qquad \text{exact sequence of graded } S\text{-modules}$$
$$e_i \mapsto x_i$$

surjective in $\deg \geq 1$

$\Downarrow$

$$\rightsquigarrow \quad 0 \to \tilde{M} \to \tilde{E} \to \tilde{S} \to 0$$

It suffices to prove that $\tilde{M} \cong \Omega_{X/Y}$.

$E_{x_i} \twoheadrightarrow S_{x_i}$ surjection of free S_{x_i}-modules

$\Rightarrow$ M_{x_i} free S_{x_i}-module of rank n.

generated by $e_j - \left(\frac{x_j}{x_i}\right) e_i$, $j \neq i$

or $\tilde{M}|_{U_i}$ generated by sections $\frac{1}{x_i} e_j - \left(\frac{x_j}{x_i^2}\right) e_i$, $j \neq i$

$$\Omega_{X/Y}|_{U_i} \xrightarrow{\cong} \tilde{M}|_{U_i}, \quad U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right]$$

$\cong$

$$\Omega_{A\left[\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right]/A}$$

$$d\left(\frac{x_j}{x_i}\right) \longmapsto \frac{1}{x_i}\underbrace{dx_j}_{e_j} - \frac{x_j}{x_i^2}\underbrace{dx_i}_{e_i}$$

generators of $\Omega_{X/Y}|_{U_i}$

it is compatible w/ the coordinate change on $U_i \cap U_j$

Theorem 2: X = irreducible separated scheme of finite type $/\mathbb{k}$. $\mathbb{k}$: algebraically closed

Then $\Omega_{X/\mathbb{k}}$ locally free of rank $n = \dim X$

iff X nonsingular variety $/\mathbb{k}$

pf: • Localization of regular local ring is a regular local ring.

It suffices to check on closed points.

$x \in X$ closed $\quad B = \mathcal{O}_{x,X}$ of dimension n

localization of k-algebra of finite type w/ residue field $\mathbb{k}$

Proposition 2 $\Rightarrow (\Omega_{X/\mathbb{k}})_x \cong \Omega_{B/\mathbb{k}}$

• B local ring. w/ residue field $\mathbb{k}$

$\mathbb{k} \subseteq B$ $\quad B$ localization of finitely generated $\mathbb{k}$-algebra

Then $\Omega_{B/\mathbb{k}}$ free B-module of rank $n \Longleftrightarrow B$ regular local ring of dimension n.

Corollary: X : variety $/k$ then $\exists\, U \underset{\text{open}}{\subseteq} X$ nonsingular

$\Omega_{X/k}$ coherent $\mathcal{O}_X$-module + generic freeness

Theorem 2: X nonsingular variety $/k$
$\cup\!|$
Y closed irreducible subscheme w/ ideal sheaf $\mathcal{J}$

Then Y nonsingular $\iff$ ① $\Omega_{Y/k}$ locally free

conormal sheaf

② $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$

exact sequence of locally free $\mathcal{O}_Y$-modules

In this case, $\mathcal{J}$ locally generated by $r = \operatorname{codim}(Y, X)$ elements

$\mathcal{J}/\mathcal{J}^2$ locally free of rank r.

pf: ($\Leftarrow$) It suffices to prove $\operatorname{rk} \Omega_{Y/k} = \dim Y$

Otherwise, $\operatorname{rk} \Omega_{Y/k} = q$

② $\Rightarrow$ $\mathcal{J}/\mathcal{J}^2$ locally free of rank $n-q$

$\underset{\text{Nakayama's lemma}}{\Longrightarrow}$ $\mathcal{J}$ can be generated by $n-q$ elements

$\Longrightarrow$ $\dim Y \geq n-(n-q) = q$

Lemma: B local ring w/ residue field k
$\cup\!|$
k then $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow[\cong]{\delta} \Omega_{B/k} \otimes k$

pf: $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \longrightarrow \cancel{\Omega_{k/k}}^{\,0} \longrightarrow 0 \implies \delta$ surjective

Claim: $\mathrm{Hom}_k(\Omega_{B/k}\otimes k, k) \xrightarrow{\delta^\vee} \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$ surjective

$$\begin{array}{ccc}
\mathrm{Hom}_k(\Omega_{B/k}\otimes k, k) & & \\
\cong & & \\
\mathrm{Hom}_B(\Omega_{B/k}, k) & & \\
\cong & & \\
\mathrm{Der}_k(B, k) & \longrightarrow & \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k) \\
\ni d & \longmapsto & \ni (d|_{\mathfrak{m}})
\end{array}$$

$\because d(\mathfrak{m}^2) = 0$

universal property of $\Omega_{B/k}$

Given $h \in \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$

$b \in B \rightsquigarrow b = \underset{k}{\lambda} + \underset{\mathfrak{m}}{c}$ define $d(\bar{b}) = h(\bar{c})$

d is a derivation
$$d(bb') = h(\overline{\lambda' c + \lambda c'}) = \lambda' h(\bar{c}) + \lambda h(\bar{c}') = b' h(\bar{c}) + b h(\bar{c}')$$

$\delta^\vee(d) = h$

$y \in Y$ closed points, $q \underset{\text{Lemma}}{=} \dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim Y$

$(A, \mathfrak{m})$ local ring, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$

$\implies q = \dim Y$

$(\Leftarrow)$ Y nonsingular $\underset{\text{Theorem 2}}{\implies}$ $\Omega_{Y/k}$ locally free of rank $\dim Y = q$

Proposition 4 $\implies$ $\mathcal{J}/\mathcal{J}^2 \xrightarrow{\delta} \Omega_{X/k}\otimes \mathcal{O}_Y \xrightarrow{\varphi} \Omega_{Y/k} \longrightarrow 0$

$y \in Y$ closed point, $\dim_k \operatorname{Ker} \varphi_y = n - q$

Choose $x_1, \dots, x_{n-q} \in \mathcal{J}$ in a neighborhood of y

s.t $dx_1, \cdots, dx_{n-q}$ generate $\ker \varphi_y$

locally free near y

$\mathcal{J}' = (x_1, \cdots, x_{n-q}) \subseteq \mathcal{J} \rightsquigarrow$ closed subscheme $Y' \ni y$

Proposition 4 $\Longrightarrow$ $\mathcal{J}'/\mathcal{J}'^2 \xrightarrow{\delta'} \Omega_{X/k}|_{Y'} \to \Omega_{Y'/k} \to 0$

$\to (dx_1, \cdots, dx_{n-q})$ locally free $\Rightarrow \Omega_{Y'/k}$ locally free of rank $n-(n-q)=q$

first part of Theorem 2 $\Downarrow$ Y': nonsingular

$Y \supseteq Y'$ integral scheme $\Rightarrow Y = Y'$ & $\mathcal{J} = \mathcal{J}'$

Definition: X nonsingular variety $/k$ of dimension n

$\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ tangent sheaf

$\omega_X := \bigwedge^n \Omega_{X/k}$ canonical sheaf

Theorem 2 $\Longrightarrow$ Y: nonsingular in X

- $0 \to \mathcal{T}_Y \to \mathcal{T}_X \to N_{Y/X} \to 0$, $\quad N_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Y)$
- $\omega_Y = \omega_X|_Y \otimes \bigwedge^r N_{Y/X}$, $r = \operatorname{codim}(Y, X)$

$r=1$, $\omega_Y = \omega_X|_Y \otimes N_{Y/X}$, $\quad N_{Y/X} \cong \mathcal{O}_X(Y)|_Y$

adjunction formula $\quad \mathcal{J}_Y = \mathcal{O}_X(-Y)$

$\mathcal{J}_Y/\mathcal{J}_Y^2 = \mathcal{O}_X(-Y)|_Y$

Theorem 1 $\Longrightarrow$ $0 \to \Omega_{\mathbb{P}^n/k} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$

$\therefore \omega_{\mathbb{P}^n_k} \cong \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$

Definition: X projective, nonsingular $/k$

$$P_g := \dim \Gamma(X, \omega_X) \quad \text{geometric genus}$$

Theorem 3. X, X' birational equivalent nonsingular projective variety $/k$

$$\Longrightarrow P_g(X) = P_g(X') \quad \text{birational invariant}$$

pf: X, X' birational $\Longleftrightarrow$

$$\begin{tikzcd} X \supseteq_{\text{open}} U \arrow[r, "\cong"] \arrow[d, phantom, "\cap"] & U' \subseteq_{\text{open}} X' \\ V \arrow[ur, "f"] & \end{tikzcd}$$

maximal open set extending $U \longrightarrow X'$

$f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}$ both locally free of rank $n = \dim X$

$\rightsquigarrow f^*\omega_{X'} \longrightarrow \omega_V$ taking $\overset{n}{\wedge}$

$$\rightsquigarrow \begin{tikzcd} \Gamma(X', \omega_{X'}) \arrow[r, "f^*"] \arrow[d] & \Gamma(V, \omega_V) \arrow[d] \\ \Gamma(U', \omega_{X'}) & \Gamma(U, \omega_V) \arrow[l, "\cong"'] \end{tikzcd}$$

$\Gamma(X', \omega_{X'}) \ni s$

$f^* s = 0 \Rightarrow s|_{U'} = 0$

$\Rightarrow s = 0 \because U'$ dense $\subseteq X'$

or f^* injective

$\therefore \dim_k \Gamma(X', \omega_{X'}) \leq \dim_k \Gamma(V, \omega_V)$

$\updownarrow$??

$\dim_k \Gamma(X, \omega_X)$

Notice that $X \setminus V$ has codim ≥ 2

$p \in X$ of codim $1 \Rightarrow \mathcal{O}_{p,X}$ discrete valuation ring

K = function field of X

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr] & V \subseteq X \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{p,X} \arrow[r, dashed, "\exists!"'] \arrow[dr] & X' \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

$\eta \in$ generic point, $\eta' \in X'$

$\mathcal{O}_{p,X} \leftarrow k$

valuation criterion of properness

projective $\Rightarrow$ proper

Similarly, $\operatorname{Spec} \mathcal{O}_{p,X} \longrightarrow X$

Thus,

$$\begin{tikzcd} V \arrow[r] & X' \\ \operatorname{Spec} K \arrow[u] \arrow[r] & \operatorname{Spec} \mathcal{O}_{p,X} \arrow[u] \end{tikzcd}$$

$V \subseteq X$ extends over $\operatorname{Im}(\operatorname{Spec} \mathcal{O}_{p,X} \to X)$

$X \supseteq \operatorname{Spec} A \ni \mathfrak{p}$, (affine open) $\quad \operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} B \subseteq Y$

$A_{\mathfrak{p}} \leftarrow B$, $\exists A_f$ $\quad \because B$ of finite type

Claim: The restriction $\Gamma(X, \omega_X) \xrightarrow{\cong} \Gamma(V, \omega_V)$

Then we have $P_g(X') \leq P_g(X)$ $\quad$ similarly $P_g(X) \leq P_g(X')$

Claim $\Leftarrow \begin{cases} \Gamma(U', \omega_U) \xrightarrow{\cong} \Gamma(U, \omega_{U'}) & \text{for } U' \subseteq U \\ \Gamma(U, \omega_U) \xrightarrow{\cong} \Gamma(\underbrace{U \cap V}_{\subseteq U \text{ complement codim} \geq 2}, \omega_{U \cap V}) . & \forall\, U \subseteq X \text{ affine open}, \text{ w/ } \omega_U \cong \mathcal{O}_U \end{cases}$

A: Noetherian integrally closed domain

then $\displaystyle\bigcap_{\mathfrak{p} : \operatorname{ht}(\mathfrak{p}) = 1} A_{\mathfrak{p}} = A$

"Hartog extension theorem" functions defined up to codimension 2 extend to global.

Remark: Plurigenera $H^0(X, K_X^d)$ is also a birational invariant.

Theorem (Bertini's theorem)

X = nonsingular closed variety of $\mathbb{P}^n_k$, k: algebraically closed

then $\exists$ hyperplane $H \subseteq \mathbb{P}^n_k$ s.t. $H \cap X$ regular.

actually true for generic element in $|H|$

pf: $x \in X$ closed points

$\rightsquigarrow B_x = \{ H \in \mathbb{P}(\underbrace{H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(1))}_{= V}) \mid H \supseteq X \text{ or } x \in H,\ x \text{ NOT regular in } X \cap H \}$

Fix $f_0 \in V$, $H_0 = \{f_0 = 0\}$

For any $f \in V$, $\frac{f}{f_0}$ defines a regular function on $\mathbb{P}^n_k \setminus H_0$

$\rightsquigarrow$: f' on $X \setminus X \cap H_0$

$x \notin H_0$, define $\varphi_x : V \longrightarrow \mathcal{O}_{x,X}/m_x^2$

$f \longmapsto$ image of f' in $\mathcal{O}_{x,X}/m_x^2$

Then • $x \in H$ iff $\varphi_x(f) \in m_x$

• x non-regular in $X \cap H$ iff $\varphi_x(f) \in m_x^2$

"H tangent to X at x"

$\varphi_x(f) \in m_x \setminus m_x^2$, $\Longrightarrow$ $\varphi_x(f) \neq$ unit, zero divisor in $\mathcal{O}_{x,X}$

$\underset{\text{Krull's Hauptidealsatz}}{\Longrightarrow}$ $ht(\mathfrak{p}) = 1$, if $\mathfrak{p} \ni \varphi_x(f)$

$\dim(\mathcal{O}_{x,X}/\varphi_x(f)) = \dim \mathcal{O}_{x,X} - 1$

Conversely, $\varphi_x(f) \in m_x^2$, $\mathcal{O}_{x,X}/(\varphi_x(f))$ cannot be regular

Otherwise $\dim(\mathcal{O}_{x,X}/(\varphi_x(f))) = \dim_k\left(\dfrac{m_x/(\varphi_x(f))}{(m_x/(\varphi_x(f)))^2} \right)$

$\wedge$ $\qquad\qquad\qquad\qquad \cong$

$\dim \mathcal{O}_{x,X}$ $\qquad\qquad \dim_k (m_x/m_x^2) = \dim \mathcal{O}_{x,X}$

Therefore, $\underset{\{f=0\}}{\overset{}{H}} \in B_x$ iff $f \in \operatorname{Ker} \varphi_x$

If $\dim X = r$, then $\dim_k \mathcal{O}_{x,X}/m_x^2 = \dim_k \mathcal{O}_x/m_x + \dim_k m_x/m_x^2$
$= 1 + r$

$\dim_k V = n+1 \xRightarrow[\varphi_x \text{ surjective}]{} B_x$ linear system of dimension $\underline{n-r-1}$

Scaling

$X \hookrightarrow \mathbb{P}_k^n$ closed embedding $\therefore$ $\{H \in V \mid x \in V\} \twoheadrightarrow m_x/m_x^2$

$H \not\ni x$ gives another 1-dim'l image.

$B = \{(x, H) \in X \times |H| \mid H \in B_x\}$ closed in $X \times |H|$

$$\begin{tikzcd} & B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & {|H|} \end{tikzcd}$$

$\dim B = \dim X + \dim B_x = r + (n-r-1) = n-1$

$f: X \twoheadrightarrow Y$ proper morphism to an irreducible variety.
If all fibres are irreducible of the same dimension,
then X is irreducible of $\dim Y + \dim X_y$.

$\Longrightarrow$ B irreducible of dimension $n-1$

$\underbrace{\dim P_2(B) \leq n-1}_{\text{by definition of dim}} < \dim |H| \xRightarrow{\text{by definition of dim}} P_2(B) \subsetneq |H|$ proper Zariski closed

Choose $H \in |H| \setminus P_2(B)$

$X \cap H$ is regular at every point.