

# Lecture 13: Differentials

Note Title

10/27/2019

## Kähler differentials

$A$ : ring,  $B = A$ -algebra,  $M = B$ -module

Definition: An  $A$ -derivation of  $B$  into  $M$  is a map

$d: B \rightarrow M$  s.t. ①  $d$  is additive

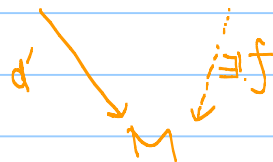
②  $d(bb') = b(d b') + b'(d b)$

③  $da = 0, \forall a \in A$

Definition:  $\Omega_{B/A} = \{db \mid b \in B\} / \sim$

module of relative differential forms of  $B$  over  $A$

$B \xrightarrow{d} \Omega_{B/A}$  w/ universal property



Proposition 1:  $0 \rightarrow I \rightarrow B \otimes_A B \xrightarrow{f} B \rightarrow 0$   
 as  $B$ -module by multiplication on the left  
 $(b, b') \mapsto bb'$

then  $I/I^2 \cong \Omega_{B/A}$ , w/  $d: B \rightarrow I/I^2$   
 $b \mapsto \underbrace{1 \otimes b - b \otimes 1}_{\text{generators of } I}$

pf:  $d: B \rightarrow I/I^2$  is a derivation  
 $b \mapsto 1 \otimes b - b \otimes 1$

$$d(bb') = 1 \otimes bb' - bb' \otimes 1 \stackrel{?}{=} b(d'b') + b'(db)$$

$$b(1 \otimes b' - b' \otimes 1) + b'(1 \otimes b - b \otimes 1)$$

$$I^2 \ni (1 \otimes b - b \otimes 1) \otimes (1 \otimes b' - b' \otimes 1) = 1 \otimes bb' + bb' \otimes 1 - b' \otimes b - b \otimes b'$$

generators of  $I^2$

$I^2 \sim d'$  satisfies Leibniz rule

ex.  $B = A[x_1, \dots, x_n]$ ,  $\Omega_{B/A} = \bigoplus_{i=1}^n B dx_i$  free  $B$ -module

② reduces  $df$  to linear combination of  $dx_i / B$

## Sheaves of Differentials

$$f: X \rightarrow Y \rightsquigarrow \Delta: X \xrightarrow{\cong} \Delta(X) \rightarrow X \times_Y X$$

closed  $\searrow$   $W$   $\nearrow$  open

$\mathcal{I} :=$  ideal sheaf of  $\Delta(X) \hookrightarrow W$

Definition:  $\Omega_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$  sheaf of relative differentials of  $X$  over  $Y$

$$U = \text{Spec } A \subseteq Y \text{ open}, \quad V = \text{Spec } B \subseteq X \text{ s.t. } f(V) \subseteq U$$

then  $V \times_U V \subseteq X \times_Y X$   $\Delta(X) \cap V \times_U V \hookrightarrow V \times_U V$

// affine open w/ ideal sheaf  $\mathcal{I}$

$$\text{Spec}(B \otimes_A B)$$

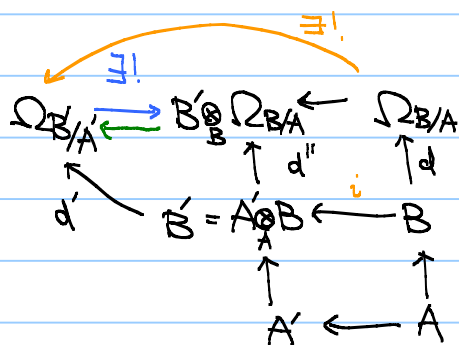
$$\mathcal{I}|_{\Delta(X) \cap V \times_U V} \cong \tilde{\mathcal{I}}$$

$$\Omega_{X/Y}|_{\Delta(X) \cap V \times_U V} \cong \tilde{\mathcal{I}/\mathcal{I}^2} \cong \tilde{\Omega_{B/A}} \quad \Omega_{X/Y} \text{ quasi-coherent}$$

Proposition 2:  $X' = X \times_Y Y' \xrightarrow{g'} X$   
 $\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$  then  $\Omega_{X'/Y'} = g'^* \Omega_{X/Y}$

pf: (Local version)  $A', B$   $A$ -algebras  $B' = A' \otimes_A B$

then  $\Omega_{B'/A'} = \Omega_{B/A} \otimes B'$



$M: B$ -module,  $M': B'$ -module

$$\text{Hom}_B(M, M') = \text{Hom}_{B'}(M \otimes_B B', M')$$

$B \xrightarrow{i} B' \rightarrow \Omega_{B'/A'}$  is a  $A$ -derivation

$B' \rightarrow B' \otimes_B \Omega_{B/A}$  is a  $A'$ -derivation

$$d''(a' \otimes 1) = a' \otimes d(\underset{A}{1}) = 0$$

$$\begin{aligned} d''((a'_1 \otimes b_1) \cdot (a'_2 \otimes b_2)) &= d''(a'_1 a'_2 \otimes b_1 b_2) = a'_1 a'_2 \otimes d(b_1 b_2) \\ &= a'_1 a'_2 \otimes (b_2 db_1 + b_1 db_2) \\ &= \underline{(a'_1 \otimes b_1) \otimes a'_2 db_2} + (a'_2 \otimes b_2) \otimes a'_1 db_1 \end{aligned}$$

In particular,  $S$  multiplicative system of  $B$

$$S^{-1} \Omega_{B/A} \cong S^{-1} A \otimes \Omega_{B/A} \cong \Omega_{S^{-1} B/A}$$

$$d\left(\frac{b}{s}\right) = \frac{db}{s} - \frac{b ds}{s^2}$$

Proposition 3.  $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\Rightarrow f^* \Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \rightarrow 0$$

pf: (local version)  $A \rightarrow B \xrightarrow{\gamma} C$  ring homomorphism

$$\text{then } \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$$

$$d_{B/A}(b) \otimes c \mapsto c d_{C/A}(\gamma(b))$$

$$d_{C/A}(c) \mapsto d_{C/B}(c)$$

Proposition 4.  $f: X \rightarrow Y$

$\downarrow$  closed w/ ideal sheaf  $\mathcal{I}$   
 $Z$

$$\text{then } \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \rightarrow \mathcal{O}_{Z/Y} \rightarrow 0$$

pf (local version)  $B: A$ -algebra,  $I \triangleleft B$ ,  $C = B/I$

$$\text{then } \frac{I}{I^2} \xrightarrow{\delta} \Omega_{B/A} \otimes C \rightarrow \Omega_{C/A} \rightarrow 0 \quad \text{as } C\text{-module}$$

naturally  $C$ -module

$$b \mapsto d_{B/A}(b) \otimes 1$$

$$d_{B/A}(b) \otimes c \mapsto c d_{C/A}(b+I)$$

Theorem 1. (Euler sequence)

$$Y = \text{Spec } A, \quad X = \mathbb{P}_A^n$$

$$\Rightarrow 0 \rightarrow \Omega_{X/Y} \rightarrow \mathcal{O}_X(-1)^{\oplus n+1} \rightarrow \mathcal{O}_X \rightarrow 0$$

dually,  $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}(1)^{\oplus n+1} \rightarrow T_{X/Y} \rightarrow 0$

relative tangent sheaf

In case of  $\mathbb{P}^1$ ,  $0 \rightarrow \Omega_{\mathbb{P}^1} \xrightarrow{\text{u1} \quad \text{deg } \Omega_{\mathbb{P}^1} = -2} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$

pf:  $S = A[x_0, \dots, x_n]$ ,  $E = S(-1)^{\oplus n+1}$   
 w/ basis  $e_0, \dots, e_n$  of degree 1  
 i.e.  $e_i = (0, \dots, \underset{i\text{-th}}{1}, \dots, 0)$

$$0 \rightarrow M \rightarrow E \rightarrow S$$

$e_i \mapsto x_i$

exact sequence of graded  $S$ -modules

surjective in  $\text{deg} \geq 1$

$$\rightsquigarrow 0 \rightarrow \tilde{M} \rightarrow \tilde{E} \rightarrow \tilde{S} \rightarrow 0$$

It suffices to prove that  $\tilde{M} \cong \Omega_{X/Y}$ .

$$E_{x_i} \twoheadrightarrow S_{x_i} \quad \text{surjection of free } S_{x_i}\text{-modules}$$

$\Rightarrow M_{x_i}$  free  $S_{x_i}$ -module of rank  $n$ .  
 generated by  $e_j - \left(\frac{x_j}{x_i}\right) e_i$ ,  $j \neq i$

or  $\tilde{M}|_{U_i}$  generated by sections  $\frac{1}{x_i} e_j - \left(\frac{x_j}{x_i}\right) e_i$ ,  $j \neq i$

$$\Omega_{X/Y}|_{U_i} \xrightarrow{\cong} \tilde{M}|_{U_i}, \quad U_i = \text{Spec } A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

$$\Omega_{A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]/A}$$

$$d\left(\frac{x_j}{x_i}\right) \longmapsto \frac{1}{x_i} dx_j - \frac{x_j}{x_i^2} dx_i$$

generators of  $\Omega_{X/k}|_{U_i}$

it is compatible w/ the coordinate change on  $U_i \cap U_j$

Theorem 2:  $X =$  irreducible separated scheme of finite type /  $k$ .  $k$ : algebraically closed

Then  $\Omega_{X/k}$  locally free of rank  $n = \dim X$

iff  $X$  nonsingular variety /  $k$

pf: • Localization of regular local ring is a regular local ring

It suffices to check on closed points.

$x \in X$  closed  $B = \mathcal{O}_{x,X}$  of dimension  $n$   
localization of  $k$ -algebra of finite type  
w/ residue field  $k$

Proposition 2  $\Rightarrow (\Omega_{X/k})_x \cong \Omega_{B/k}$

•  $B$  local ring w/ residue field  $k$

$B$  localization of finitely generated  $k$ -algebra

Then  $\Omega_{B/k}$  free  $B$ -module of rank  $n \iff B$  regular local ring of dimension  $n$ .

Corollary:  $X$ : variety  $/\mathbb{k}$  then  $\exists U \subseteq_{\text{open}} X$  nonsingular

$\Omega_{X/\mathbb{k}}$  coherent  $\mathcal{O}_X$ -module + generic freeness

Theorem 2:  $X$  nonsingular variety  $/\mathbb{k}$

$U \downarrow$   
 $Y$  closed irreducible subscheme w/ ideal sheaf  $\mathcal{I}$

Then  $Y$  nonsingular  $\iff \mathcal{O}_Y \otimes \Omega_{X/\mathbb{k}}$  locally free

conormal sheaf

$$\textcircled{2} \quad 0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/\mathbb{k}} \otimes \mathcal{O}_Y \rightarrow \Omega_{Y/\mathbb{k}} \rightarrow 0$$

exact sequence of locally free  $\mathcal{O}_Y$ -modules

In this case,  $\mathcal{I}$  locally generated by  $r = \text{codim}(Y, X)$  elements

$\mathcal{I}/\mathcal{I}^2$  locally free of rank  $r$ .

pf:  $(\Leftarrow)$  It suffices to prove  $\text{rk } \Omega_{Y/\mathbb{k}} = \dim Y$

Otherwise,  $\text{rk } \Omega_{Y/\mathbb{k}} = q$

$\textcircled{2} \Rightarrow \mathcal{I}/\mathcal{I}^2$  locally free  
of rank  $n-q$

$\implies \mathcal{I}$  can be generated by  $n-q$  elements

Nakayama's lemma

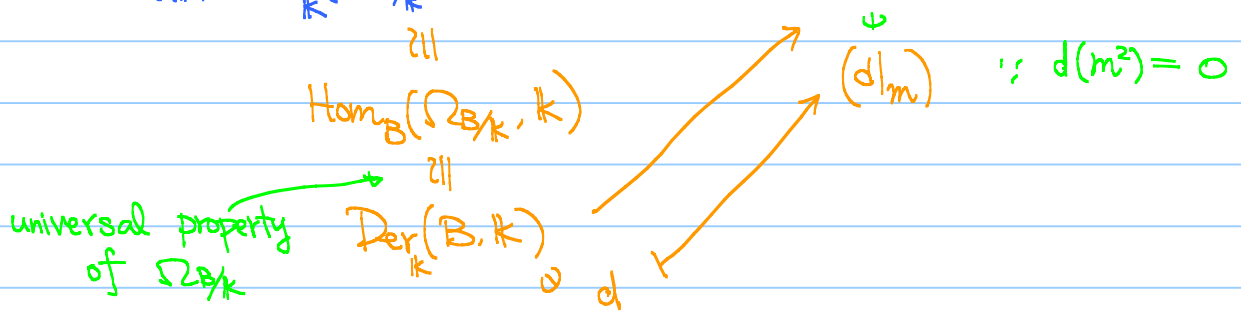
$\implies \dim Y \geq n - (n-q) = q$

Lemma:  $B$  local ring w/ residue field  $k$

$U \downarrow$   
 $k$  then  $\mathfrak{m}^2 \xrightarrow{\cong} \Omega_{B/k} \otimes k$

pf:  $m/m^2 \rightarrow \Omega_{B/k} \rightarrow \Omega_{k/k} \rightarrow 0 \Rightarrow \delta$  surjective

Claim:  $\text{Hom}_k(\Omega_{B/k}, k) \xrightarrow{\delta^*} \text{Hom}(m/m^2, k)$  surjective



Given  $\tilde{h} \in \text{Hom}(m/m^2, k)$

$b \in B \rightsquigarrow b = \lambda + c$  define  $d(b) = \tilde{h}(c)$   
 $\quad \quad \quad \uparrow \quad \quad \uparrow$   
 $\quad \quad \quad k \quad \quad m$

$d$  is a derivation

$\delta^*(d) = \tilde{h}$

$d(b'b) = \tilde{h}(\lambda c' + \lambda' c)$

$= \lambda \tilde{h}(c) + \lambda' \tilde{h}(c')$

$= b \tilde{h}(c) + b' \tilde{h}(c')$

$y \in Y$  closed points,  $q = \dim_k(m/m^2) \geq \dim Y$   
 Lemma  $\uparrow$

(A,m) local ring,  $\dim m/m^2 \geq \dim A$

$\Rightarrow q = \dim Y$

$(\Rightarrow) Y$  nonsingular  $\Rightarrow \Omega_{Y/k}$  locally free of rank  $\dim Y = q$   
 Theorem 2

Proposition 4  $\Rightarrow \mathcal{O}_Y \xrightarrow{\delta} \Omega_{Y/k} \otimes \mathcal{O}_Y \xrightarrow{\varphi} \Omega_{Y/k} \rightarrow 0$

$y \in Y$  closed point,  $\dim_k \ker \varphi_y = n - q$

Choose  $x_1, \dots, x_{n-q} \in \mathcal{O}_Y$  in a neighborhood of  $y$



s.t.  $dx_1, \dots, dx_{n-q}$  generate  $\ker \varphi_y$

locally free near  $y$

$\mathcal{J}' = (x_1, \dots, x_{n-q}) \subseteq \mathcal{J} \rightsquigarrow$  closed subscheme  $Y' \subseteq Y$

Proposition 4  $\Rightarrow \mathcal{J}'/\mathcal{J}'^2 \xrightarrow{\sigma'} \Omega_{X/k}|_{Y'} \rightarrow \Omega_{Y'/k} \rightarrow 0$

$(dx_1, \dots, dx_{n-q}) \Rightarrow \Omega_{Y'/k}$  locally free of rank  $n - (n-q) = q$

first part of Theorem 2  $\Downarrow$   
 $Y$ : nonsingular

$Y \supseteq Y'$  integral scheme  $\Rightarrow Y = Y'$  &  $\mathcal{J} = \mathcal{J}'$

Definition:  $X$  nonsingular variety /  $k$  of dimension  $n$

$\mathcal{T}_X := \text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$  tangent sheaf

$\omega_X := \bigwedge^n \Omega_{X/k}$  canonical sheaf

Theorem 2  $\Rightarrow Y$ : nonsingular in  $X$

$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_X \rightarrow \mathcal{N}_{Y/X} \rightarrow 0$

$\omega_Y = \omega_X|_Y \otimes \bigwedge^r \mathcal{N}_{Y/X}, r = \text{codim}(Y, X)$

$r=1, \omega_Y = \omega_X|_Y \otimes \mathcal{N}_{Y/X} \cong \mathcal{O}_X(Y)|_Y$

adjunction formula  $\mathcal{J}_Y = \mathcal{O}_X(-Y)$

$\mathcal{J}'/\mathcal{J}'^2 = \mathcal{O}_X(-Y)|_Y$

Theorem 1  $\Rightarrow 0 \rightarrow \Omega_{\mathbb{P}^n/k} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0$

$\therefore \omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$

Definition:  $X$  projective, nonsingular  $/\mathbb{k}$

$$P_g := \dim T(X, \omega_X) \text{ geometric genus}$$

Theorem 3.  $X, X'$  birational equivalent nonsingular projective variety  $/\mathbb{k}$

$$\Rightarrow P_g(X) = P_g(X') \text{ birational invariant}$$

pf:  $X, X'$  birational  $\Leftrightarrow$

$$\begin{array}{ccc} X \supseteq U & \xrightarrow{\cong} & U' \subseteq X' \\ \text{open} & \parallel & \text{open} \\ & \searrow f & \\ & V & \end{array}$$

maximal open set extending  $U \rightarrow X$

$$f^* \Omega_{X'/\mathbb{k}} \rightarrow \Omega_{V/\mathbb{k}} \text{ both locally free of rank } n = \dim X$$

$$\rightsquigarrow f^* \omega_{X'} \rightarrow \omega_V \text{ taking } \wedge$$

$$\rightsquigarrow T(X, \omega_X) \xrightarrow{f^*} T(V, \omega_V)$$

$$f^* s = 0 \Rightarrow s|_V = 0$$

$$\Rightarrow s = 0 \because U \text{ dense} = X'$$

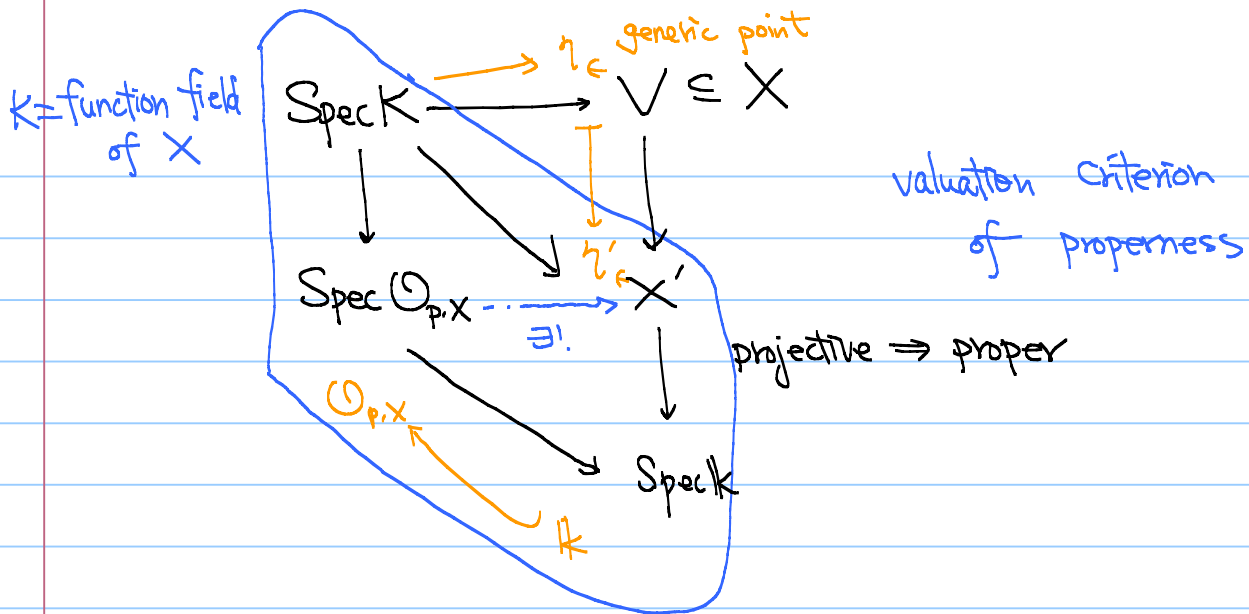
or  $f^*$  injective

$$\therefore \dim_{\mathbb{k}} T(X, \omega_X) \leq \dim_{\mathbb{k}} T(V, \omega_V)$$

$$\begin{array}{c} \updownarrow ?? \\ \dim_{\mathbb{k}} T(X, \omega_X) \end{array}$$

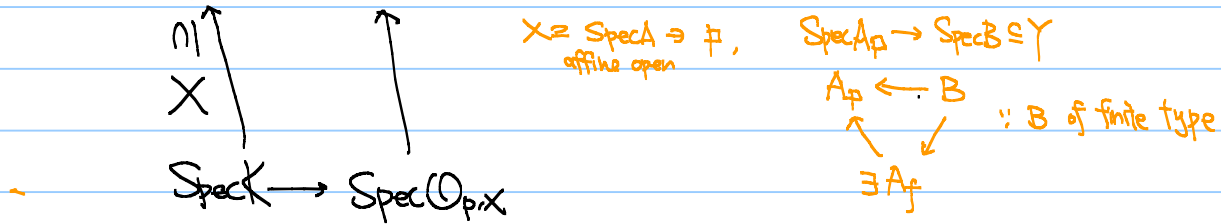
Notice that  $X \setminus V$  has  $\text{codim} \geq 2$

$p \in X$  of  $\text{codim} 1 \Rightarrow \mathcal{O}_{p,X}$  discrete valuation ring



Similarly,  $\text{Spec } \mathcal{O}_{p,X} \rightarrow X$

Thus,  $V \rightarrow X'$  extends over  $\text{Im}(\text{Spec } \mathcal{O}_{p,X} \rightarrow X)$



Claim: The restriction  $\Gamma(X, \omega_X) \xrightarrow{\cong} \Gamma(V, \omega_V)$

Then we have  $P_g(X') \leq P_g(X)$  similarly  $P_g(X) \leq P_g(X')$

Claim  $\Leftarrow \begin{cases} \Gamma(U, \omega_U) \cong \Gamma(U, \omega_U) \text{ for } U \subseteq U \\ \Gamma(U, \omega_U) \cong \Gamma(\underbrace{U \cup V}_{\substack{\cap \text{ complement} \\ \text{codim} \geq 2}}, \omega_{U \cup V}), \forall U \subseteq X, \omega_U \cong \mathcal{O}_U \\ \text{affine open} \end{cases}$

A: Noetherian integrally closed domain

then  $\bigcap_{\# \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}} = A$

"Hartog extension theorem" functions defined up to codimension 2 extend to global.

Remark: Plurigenera  $H^0(X, K_X^d)$  is also a birational invariant.

### Theorem (Bertini's theorem)

$X =$  nonsingular closed variety of  $\mathbb{P}_K^n$ ,  $K$ : algebraically closed

then  $\exists$  hyperplane  $H \subseteq \mathbb{P}_K^n$  st  $H \cap X$  regular.  
 actually true for generic element in  $|H|$

pf:  $x \in X$  closed points

$$\leadsto B_x = \left\{ H \in \mathbb{P}(H^0(\mathbb{P}_K^n, \mathcal{O}_{\mathbb{P}^n}(1))) \mid \begin{array}{l} H \neq X \text{ or } x \in H \\ x \text{ NOT regular in } X \cap H \end{array} \right\}$$

Fix  $f_0 \in V$ ,  $H_0 = \{f_0 = 0\}$

For any  $f \in V$ ,  $\frac{f}{f_0}$  defines a regular function on  $\mathbb{P}_K^n - H_0$   
 $\rightsquigarrow = f'$  on  $X - X \cap H_0$

$x \notin H_0$ , define  $\varphi_x: V \rightarrow \mathcal{O}_{x,X}/\mathfrak{m}_x^2$

$f \mapsto$  image of  $f'$  in  $\mathcal{O}_{x,X}/\mathfrak{m}_x^2$

Then  $\bullet x \in H$  iff  $\varphi_x(f) \in \mathfrak{m}_x$

$\bullet x$  non-regular in  $X \cap H$  iff  $\varphi_x(f) \in \mathfrak{m}_x^2$   
 "H tangent to X at x"

$\varphi_x(f) \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2 \implies \varphi_x(f) \neq \text{unit}$ , zero divisor in  $\mathcal{O}_{x,X}$

$\implies \text{ht}(\mathfrak{p}) = 1$ , if  $\mathfrak{p} \ni \varphi_x(f)$   
 Krull's Hauptidealsatz

$$\dim(\mathcal{O}_{x,X}/(\varphi_x(f))) = \dim \mathcal{O}_{x,X} - 1$$

Conversely,  $\varphi_x(f) \in \mathfrak{m}_x^2$ ,  $\mathcal{O}_{x,X}/(\varphi_x(f))$  cannot be regular

otherwise 
$$\dim(\mathcal{O}_{x,X}/(\varphi_x(f))) = \dim_k \left( \frac{\mathfrak{m}_x/\varphi_x(f)}{(\mathfrak{m}_x/\varphi_x(f))^2} \right)$$

$$\uparrow \qquad \qquad \qquad \downarrow$$

$$\dim \mathcal{O}_{x,X} \qquad \qquad \qquad \dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim \mathcal{O}_{x,X}$$

Therefore,  $H \in \mathcal{B}_x$  iff  $f \in \ker \varphi_x$   
 $\{f=0\}$

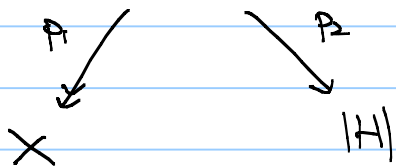
If  $\dim X = r$ , then 
$$\dim_k \mathcal{O}_{x,X}/\mathfrak{m}_x^2 = \dim_k \mathcal{O}_x/\mathfrak{m}_x + \dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$$

$$= 1 + r$$

$\dim_k V = n+1 \implies \mathcal{B}_x$  linear system of dimension  $n-r-1$   
 $\varphi_x$  surjective Scaling

$X \hookrightarrow \mathbb{P}_k^n$  closed embedding  $\dots \{H \in V \mid x \in V\} \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$   
 $H \neq x$  gives another  $\pm$  diral image.

$\mathcal{B} = \{(x, H) \in X \times |H| \mid H \in \mathcal{B}_x\}$  closed in  $X \times |H|$



$\dim \mathcal{B} = \dim X + \dim \mathcal{B}_x = r + (n-r-1) = n-1$

$f: X \rightarrow Y$  proper morphism to an irreducible variety.

If all fibres are irreducible of the same dimension,

then  $X$  is irreducible of  $\dim Y + \dim X_y$ .

$\Rightarrow B$  irreducible of dimension  $n-1$

$\dim \mathcal{P}_2(B) \leq n-1 < \dim |H| \implies \mathcal{P}_2(B) \subsetneq |H|$   
by definition of dim      by definition of dim      proper Zasliski closed

Choose  $H \in |H| \setminus \mathcal{P}_2(B)$

$X \cap H$  is regular at every point.